Exercise 1. For technical reasons, it is easier to write the conformal map as

$$f(z)=\frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}$$

where $\alpha,\beta\in\mathbb{C}$ and $|\alpha|>|\beta|$ (by dividing the numerator and denominator by $|\alpha|$, we return to the usual form). Now, the equation $f(e^{i\theta})=e^{i\varphi}$ is equivalent to

$$e^{i\theta}\alpha+\beta=e^{i(\theta+\varphi)}\overline{\alpha}+e^{i\varphi}\overline{\beta},$$

which yields, after identifying the real and imaginary parts, to

$$\begin{cases}(\cos(\theta)-\cos(\varphi))\mathrm{Re}\,(\alpha)-(\sin(\theta)+\sin(\varphi))\mathrm{Im}\,(\alpha)+(1-\cos(\theta+\varphi))\mathrm{Re}\,(\beta)-\sin(\theta+\varphi)\mathrm{Im}\,(\beta)=0\\(\sin(\theta)-\sin(\varphi))\mathrm{Re}\,(\alpha)+(\cos(\theta)+\cos(\varphi))\mathrm{Im}\,(\alpha)-\sin(\theta+\varphi)\mathrm{Re}\,(\beta)+(1+\cos(\theta+\varphi))\mathrm{Im}\,(\beta)=0.\end{cases}$$

We rewrite it in a matrix form as

$$\begin{pmatrix}\cos(\theta)-\cos(\varphi) & -(\sin(\theta)+\sin(\varphi))\\ \sin(\theta)-\sin(\varphi) & \cos(\theta)+\cos(\varphi)\end{pmatrix}\begin{pmatrix}\mathrm{Re}\,(\alpha)\\ \mathrm{Im}\,(\alpha)\end{pmatrix}+\begin{pmatrix}1-\cos(\theta+\varphi) & -\sin(\theta+\varphi)\\ -\sin(\theta+\varphi) & 1+\cos(\theta+\varphi)\end{pmatrix}\begin{pmatrix}\mathrm{Re}\,(\beta)\\ \mathrm{Im}\,(\beta)\end{pmatrix}=0.$$

Notice that both matrices are singular. Using the following sum-to-product formulae

$$\begin{cases}\sin(\theta)\pm\sin(\varphi)=2\sin\left(\dfrac{\theta\pm\varphi}{2}\right)\cos\left(\dfrac{\theta\mp\varphi}{2}\right)\\ \cos(\theta)+\cos(\varphi)=2\cos\left(\dfrac{\theta+\varphi}{2}\right)\cos\left(\dfrac{\theta-\varphi}{2}\right)\\ \cos(\theta)-\cos(\varphi)=-2\sin\left(\dfrac{\theta+\varphi}{2}\right)\sin\left(\dfrac{\theta-\varphi}{2}\right)\end{cases}$$

and the identities

$$\begin{cases}\cos(2x)=2\cos^2(x)-1=1-\sin^2(x)\\ \sin(2x)=2\cos(x)\sin(x)\end{cases}$$

the system becomes

$$\begin{pmatrix}-2\sin\left(\dfrac{\theta+\varphi}{2}\right)\sin\left(\dfrac{\theta-\varphi}{2}\right) & -2\sin\left(\dfrac{\theta+\varphi}{2}\right)\cos\left(\dfrac{\theta-\varphi}{2}\right)\\ 2\sin\left(\dfrac{\theta-\varphi}{2}\right)\cos\left(\dfrac{\theta+\varphi}{2}\right) & 2\cos\left(\dfrac{\theta+\varphi}{2}\right)\cos\left(\dfrac{\theta-\varphi}{2}\right)\end{pmatrix}\begin{pmatrix}\mathrm{Re}\,(\alpha)\\ \mathrm{Im}\,(\alpha)\end{pmatrix}$$
$$+\begin{pmatrix}2\sin^2\left(\dfrac{\theta+\varphi}{2}\right) & -2\cos\left(\dfrac{\theta+\varphi}{2}\right)\sin\left(\dfrac{\theta+\varphi}{2}\right)\\ -2\cos\left(\dfrac{\theta+\varphi}{2}\right)\sin\left(\dfrac{\theta+\varphi}{2}\right) & 2\cos^2\left(\dfrac{\theta+\varphi}{2}\right)\end{pmatrix}\begin{pmatrix}\mathrm{Re}\,(\beta)\\ \mathrm{Im}\,(\beta)\end{pmatrix}=0.$$

Notice that first line has a $\sin\left(\dfrac{\theta+\varphi}{2}\right)$ in factor and that the second line has a $\cos\left(\frac{\theta+\varphi}{2}\right)$, and that those two numbers cannot be simultaneous 0. Simplifying the first equation by $2\sin\left(\frac{\theta}{\varphi}\right)$ yields the equation

$$-\sin\left(\frac{\theta-\varphi}{2}\right)\mathrm{Re}\,(\alpha)-\cos\left(\frac{\theta-\varphi}{2}\right)\mathrm{Im}\,(\alpha)+\sin\left(\frac{\theta+\varphi}{2}\right)\mathrm{Re}\,(\beta)-\cos\left(\frac{\theta+\varphi}{2}\right)\mathrm{Im}\,(\beta)=0,$$

and simplifying the second one by $\cos\left(\dfrac{\theta+\varphi}{2}\right)$ yields

$$\sin\left(\frac{\theta-\varphi}{2}\right)\mathrm{Re}\,(\alpha)+\cos\left(\frac{\theta-\varphi}{2}\right)\mathrm{Im}\,(\alpha)-\sin\left(\frac{\theta+\varphi}{2}\right)\mathrm{Re}\,(\beta)+\cos\left(\frac{\theta+\varphi}{2}\right)\mathrm{Im}\,(\beta)=0, \tag{1}$$

Those two equations are the same, so the system reduces to the equation (1). Now, if $0\leq\theta_1<\theta_2<\theta_3<2\pi$ and $0\leq\varphi_1<\varphi_2<\varphi_3<2\pi$, it is easy to show that the matrix

$$\begin{pmatrix}\sin\left(\frac{\theta_1-\varphi_1}{2}\right) & \cos\left(\frac{\theta_1-\varphi_1}{2}\right) & -\sin\left(\frac{\theta_1+\varphi_1}{2}\right) & \cos\left(\frac{\theta_1+\varphi_1}{2}\right)\\ \sin\left(\frac{\theta_2-\varphi_2}{2}\right) & \cos\left(\frac{\theta_2-\varphi_2}{2}\right) & -\sin\left(\frac{\theta_2+\varphi_2}{2}\right) & \cos\left(\frac{\theta_2+\varphi_2}{2}\right)\\ \sin\left(\frac{\theta_3-\varphi_3}{2}\right) & \cos\left(\frac{\theta_3-\varphi_3}{2}\right) & -\sin\left(\frac{\theta_3+\varphi_3}{2}\right) & \cos\left(\frac{\theta_3+\varphi_3}{2}\right)\end{pmatrix}$$

has rank 3, which yields a solution α,β (to be precise, a one-dimensional family of solutions), and renormalising it by $|\alpha|$, it becomes unique as an element of $[0,2\pi[\times\mathbb{D}$ (written $(e^{i\psi},a)$ with $0\leq\psi<2\pi$).

Exercise 2. 1. One inequality is trivial since for all $\varphi\in C_c^\infty(\mathbb{D},\mathbb{R}^3)$ such that $\|\varphi\|_{\mathrm{L}^\infty(\mathbb{D})}\leq 1$, we have

$$\int_{\mathbb{D}}\langle\partial_x u\times\partial_y u,\varphi\rangle dx\,dy\leq\left(\int_{\mathbb{D}}|\partial_x u\times\partial_y y|dy\,dy\right)\|\varphi\|_{\mathrm{L}^\infty(\mathbb{D})}\leq\int_{\mathbb{D}}|\partial_x u\times\partial_y y|dy\,dy.$$

On the other hand, for all $0<\varepsilon<1$, regularising by convolution the bounded, compactly supported function

$$\frac{\partial_x\varphi\times\partial_y\varphi}{|\partial_x\varphi\times\partial_y\varphi|}\mathbf{1}_{\mathbb{D}(0,1-\varepsilon)}$$

by convolution, we find a sequence $\{\varphi_n\}_{n\in\mathbb{N}}\in\mathscr{D}(\mathbb{D},\mathbb{R}^3)$ such that $\|\varphi_n\|_{\mathrm{L}^\infty(\mathbb{D})}\leq 1$ and such that for all $f\in L^1(\mathbb{D},\mathbb{R}^3)$, we have

$$\int_{\mathbb{D}}\langle f,\varphi_n\rangle\,dx\,dy\underset{n\to\infty}{\longrightarrow}\int_{\mathbb{D}(0,1-\varepsilon)}\left\langle f,\frac{\partial_x\varphi\times\partial_y\varphi}{|\partial_x\varphi\times\partial_y\varphi|}\right\rangle dx\,dy.$$

Applying this to $f=\partial_x u\wedge\partial_y u$ and letting $\varepsilon\to 0$, the result follows.

2. Integrating by parts, we get

$$\int_{\mathbb{D}}\langle\partial_x u_n\times\partial_y u_n,\varphi\rangle dx\,dy=-\frac{1}{2}\int_{\mathbb{D}}\left(\langle u_n\times\partial_y u_n,\partial_x\varphi\rangle+\langle\partial_x u_n\times u_n,\partial_y\varphi\rangle\right)dx\,dy.$$

Thanks to Rellich-Kondrachov theorem, $\{u_n\}_{n\in\mathbb{N}}$ converges strongly in L^2 and since φ is smooth, a result of the course shows that

$$\int_{\mathbb{D}}\left(\langle u_n\times\partial_y u_n,\partial_x\varphi\rangle+\langle\partial_x u_n\times u_n,\partial_y\varphi\rangle\right)dx\,dy\underset{n\to\infty}{\longrightarrow}\int_{\mathbb{D}}\left(\langle u\times\partial_y u,\partial_x\varphi\rangle+\langle\partial_x u_n\times u,\partial_y\varphi\rangle\right)dx\,dy.$$

3. We have

$$\begin{aligned}\int_{\mathbb{D}}\langle\partial_x u\times\partial_y u,\varphi\rangle dx\,dy&=\lim_{n\to\infty}\int_{\mathbb{D}}\langle\partial_x u_n\times\partial_y u_n,\varphi\rangle dx\,dy\\&\leq\liminf_{n\to\infty}\sup\left\{\int_{\mathbb{D}}\langle\partial_x u_n\times\partial_x u_n,\psi\rangle dx\,dy:\ \psi\in C_c^\infty(\mathbb{D},\mathbb{R}^3),\ \|\psi\|_{\mathrm{L}^\infty(\mathbb{D})}\leq 1\right\}\\&=\liminf_{n\to\infty}A(u_n).\end{aligned}$$

Exercise 3. We know by a previous exercise (Série 6, Exercise 3) that the homogenous H^s norm squared is equivalent to

$$\int_0^{2\pi}\int_0^{2\pi}\frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}dxdy,$$

since since $\sin(x) = x + O(x^3)$, the equivalence follows easily by taking $s = 1/2$.